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The density of ones in Pascal's rhombus

John Goldwasser[a], William Klostermeyer[b,*] Michael Mays[a], George Trapp[b]

[a] *Department of Mathematics, West Virginia University, Morgantown, WV 26506-6330, USA*
[b] *Department of Computer Science and Elec. Engineering, West Virginia University, Morgantown, WV 26506-6330, USA*

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We dedicate this paper (and the rhombus) to Professor Henry Gould on the occasion of his 70th birthday and in appreciation of his work on recurrences such as the rhombus

Abstract

Pascal's rhombus is a variation of Pascal's triangle in which values are computed as the sum of four terms, rather than two. It is shown that the limiting ratio of the number of ones to the number of zeros in Pascal's rhombus, taken modulo 2, approaches zero. An asymptotic formula for the number of ones in the rhombus is also shown.

1. Introduction

Pascal's rhombus is defined in [2] as a variation on Pascal's triangle. Consider an infinite array $R=[r_{i,j}]$ where i is a non-negative integer and j is an integer, $r_{i,j}$ defined by

$$\begin{aligned} &r_{0,j}=0 \text{ for all } j; \quad r_{1,j}=0 \text{ for all } j\neq 0; \quad r_{1,0}=1, \\ &r_{i+1,j}=r_{i,j}+r_{i,j-1}+r_{i-1,j}+r_{i,j+1} \quad \text{for } i\geqslant 1. \end{aligned} \tag{1}$$

Note that $r_{i,j}=0$ if $|j|\geqslant i$. We now define Pascal's rhombus to be the sub-array of R where $i\geqslant 0$ and for a fixed value of i we have $-i\leqslant j\leqslant i$. The first few rows of the rhombus and the rhombus (mod 2) are shown in Figs. 1 and 2.

Several properties of the rhombus and the related *left-bounded rhombus* (in which the rhombus 'grows' infinitely only to the right) are given in [2]. Our interest in the rhombus initially stems from [1], in which the same recurrence (mod 2) arose in the

* Corresponding author. E-mail: wfk@cs.wvu.edu.

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$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & 1 & 1 & & & \\ & & 1 & 2 & 4 & 2 & 1 & & \\ & 1 & 3 & 8 & 9 & 8 & 3 & 1 & \\ 1 & 4 & 13 & 22 & 29 & 22 & 13 & 4 & 1 \end{array}$$

Fig. 1. First five rows of Pascal's rhombus (zeroes not shown).

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & 1 & 1 & & & \\ & & 1 & 0 & 0 & 0 & 1 & & \\ & 1 & 1 & 0 & 1 & 0 & 1 & 1 & \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}$$

Fig. 2. Pascal's rhombus (mod 2).

study of a problem concerning grid graphs, though the array there is bounded on both the left and right. The following conjecture is from [2].

Conjecture. Let a_n and f_n be the number of ones and zeroes, respectively, in the first n rows of Pascal's rhombus (mod 2). Then $\lim_{n\to\infty} a_n/f_n = 0$.

A computer program verified that in the first 100 000 rows of the rhombus (mod 2), 10.2% of the entries are 1's. Thus, the ratio of ones to zeroes does not seem to approach zero very quickly! In this paper we will first prove a result which might lead one to suspect the conjecture is false, but then we prove that the conjecture is, in fact, true by obtaining a formula for the number of ones in the first 2^k rows of Pascal's rhombus (mod 2).

2. Results

Let $\mathscr{A} = [a_{i,j}]$ be the binary array obtained by reading Pascal's rhombus mod 2. The recurrence relations (all mod 2) may be re-written as

$$\begin{aligned} &a_{0,j} = 0 \text{ for all } j; \qquad a_{1,j} = 0 \text{ for all } j \neq 0; \qquad a_{1,0} = 1, \\ &a_{i,j} = a_{i-1,j} + a_{i+1,j} + a_{i,j+1} + a_{i,j-1} \quad (i \geqslant 1), \end{aligned} \tag{2}$$

where we have rearranged the terms from Eq. (1) to get (2). If the closed neighborhood of an entry includes itself and the four immediately adjacent neighbors (two in the same row, two in the same column) then the sum of the entries in the closed neighborhood of $a_{i,j}$ is zero, for all $i \geqslant 1$ and all j.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & 0 & 1 & & \\ & 1 & 1 & 1 & 1 & 1 & \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{array} \qquad \begin{array}{cccccccc} & & & 1 & 1 & & & \\ & & 1 & 0 & 0 & 1 & & \\ & 1 & 1 & 0 & 0 & 1 & 1 & \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \end{array}$$

Fig. 3. Sub-arrays $\mathscr{B}$ and $\mathscr{C}$.

If $i \geqslant 2$ then we get relations like (2) for each of the four summands on the right-hand side of (2). Substituting this into (2) gives

$$a_{i,j} = a_{i-2,j} + a_{i+2,j} + a_{i,j-2} + a_{i,j+2} \quad (i \geqslant 2). \tag{3}$$

Repeating this procedure gives

$$a_{i,j} = a_{i-2^t,j} + a_{i+2^t,j} + a_{i,j-2^t} + a_{i,j+2^t} \quad (i \geqslant 2^t) \tag{4}$$

for all non-negative integers t.

We partition the array $\mathscr{A} = [a_{i,j}]$ into four sub-arrays, the first taking all entries $a_{i,j}$ where i and j are both even, the second where i is odd and j is even, the third where i is even and j is odd, and the fourth where both i and j are odd. We denote these subarrays by $\mathscr{E}, \mathscr{B}, \mathscr{C}$, and $\mathscr{D}$, respectively. Because of Eq. (3), the entries of the four sub-arrays can be easily generated once the first two rows are known. It turns out that $\mathscr{B}$ and $\mathscr{C}$ are as shown in Fig. 3, $\mathscr{E} = \mathscr{A}$, and all entries in $\mathscr{D}$ are zero — this is because all entries in the first two rows of $\mathscr{D}$ are equal to zero, since $a_{1,j} = a_{3,j} = 0$ for all odd j, and thus all subsequent rows of $\mathscr{D}$ will contain all zeroes, by Eq. (3) which implies that entries in $\mathscr{D}$ obey a recurrence identical to (2). Likewise, from Eq. (3) it is easily seen that $\mathscr{E} = \mathscr{A}$.

Let A_n, B_n, C_n be the number of ones in the nth row of the sub-arrays $\mathscr{A}, \mathscr{B}$ and $\mathscr{C}$, respectively. Similar to the partitioning of $\mathscr{A}$, the array $\mathscr{C}$ can be partitioned into four sub-arrays based on the parities of the indices. Each such sub-array will obey a recurrence identical to (2) once the first two rows are determined. From Fig. 3 we see that row 1 of $\mathscr{C}$ contains the string 11, row 2 contains 1001, etc. Considering these sub-arrays of $\mathscr{A}$ and $\mathscr{C}$, for any positive integer t we obtain

$$A_{2t} = A_t + C_t, \tag{5}$$

$$C_{2t} = 2A_t, \tag{6}$$

$$C_{2t-1} = 2B_t = 2A_{2t-1}. \tag{7}$$

Let p be an odd positive integer and i be an integer greater than 1. From equalities (5) and (6) we get

$$A_{p2^i} = A_{p2^{i-1}} + 2A_{p2^{i-2}} \tag{8}$$

and from Eqs. (5) and (7) we get

$$A_{2p} = 3A_p. \tag{9}$$

The solution to the recurrence given by Eqs. (8) and (9) is

$$A_{p2^i} = \frac{2^{i+2}-(-1)^i}{3}A_p, \quad i=0,1,2,\ldots. \tag{10}$$

Theorem 1. *Let A_n be the number of ones in the* nth *row of Pascal's rhombus* (*mod* 2) *and let $\delta(n)=A_n/(2n-1)$ be the fraction of the entries in the* nth *row that are ones. If $n=p2^i$ where p is an odd integer, then*

$$A_{p2^i} = \frac{2^{i+2}-(-1)^i}{3}A_p, \quad i=0,1,2,\ldots$$

and

$$\lim_{i\to\infty} \delta(p2^i) = \frac{4}{3}\frac{2p-1}{2p}\delta(p).$$

Proof. The first equation was proved above. It follows that

$$\delta(p2^i) = \delta(p)\frac{4}{3}\frac{2p-1}{2p}\frac{p(2^{i+2}-(-1)^i)}{p2^{i+2}-2} \tag{11}$$

which proves the second equation. □

Theorem 1 affirms Conjecture 2 from [2], which concerned the special case when $p=1$.

It is an obvious consequence of Theorem 1 that the sequence $\mathscr{T}=\{\delta(1),\delta(2),\delta(3),\ldots\}$ diverges. Using Eqs. (5)–(7), similar formulas for A_{2t+1}, B_{2t} and B_{2t+1} along with Eq. (11), it is not hard to show that the lim sup of $\mathscr{T}$ is $\frac{2}{3}$, that $\lim_{i\to\infty}\delta(p2^i)=\frac{2}{3}$ if and only if $p=1$ or $p=5$, and that $\delta(n)\geqslant\frac{2}{3}$ if and only if $n=2^i$ where i is any non-negative integer or $n=5\times 2^i$ where i is odd.

In spite of Theorem 1, we will prove that the conjecture is correct. Let a_t, b_t, c_t be the number of 1's in the first t rows of $\mathscr{A}, \mathscr{B}$, and $\mathscr{C}$, respectively. For example, $b_1=1$, $b_2=3$, $b_3=8$. Again, because of the partitions of $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ into four sub-arrays, we get

$$a_n = b_{\lceil n/2\rceil} + a_{\lfloor n/2\rfloor} + c_{\lfloor n/2\rfloor}, \tag{12}$$

$$b_n = a_{\lceil n/2\rceil} + a_{\lfloor n/2\rfloor-1} + c_{\lfloor n/2\rfloor} + c_{\lceil n/2\rceil-1} \tag{13}$$

and

$$c_n = 2b_{\lceil n/2\rceil} + 2a_{\lfloor n/2\rfloor}. \tag{14}$$

From (12) and (14) we get

$$a_n = \tfrac{1}{2}c_n + c_{\lfloor n/2 \rfloor}. \tag{15}$$

If $n=8t$, where t is a positive integer, then from (13) and (14) we get

$$\begin{aligned} c_{8t} &= 2a_{2t} + 2a_{2t-1} + 2c_{2t} + 2c_{2t-1} + 2a_{4t} \\ &= 2a_{4t} + 4a_{2t} + 4c_{2t} - 2A_{2t} - 2C_{2t} \end{aligned} \tag{16}$$

and then from (6), (15), and (16)

$$a_{16t} = a_{8t} + 8a_{4t} + 4a_{2t} - A_{4t} - 4A_{2t} - 4A_t. \tag{17}$$

If $t = p\cdot 2^i$ where p is an odd integer, then by (10)

$$\begin{aligned} A_{4t} + 4A_{2t} + 4A_t &= \frac{A_p}{3}[(2^{i+4}-(-1)^i) + 4(2^{i+3}+(-1)^i) + 4(2^{i+2}-(-1)^i)] \\ &= \frac{2^{i+6}-(-1)^i}{3}A_p, \end{aligned} \tag{18}$$

so from (17) and (18) we get

$$a_{16t} = a_{8t} + 8a_{4t} + 4a_{2t} - \frac{2^{i+6}-(-1)^i}{3}A_p. \tag{19}$$

Now we define a sequence $\{d_i\}$ by $d_i = a_{2^i}$ $(i=0,1,2,\ldots)$. Since $a_1=1$, $a_2=4$, $a_4=11$ and $a_8=36$, by letting $p=1$ in (19) we get the following recurrence relation for $\{d_i\}$:

$$\begin{gathered} d_i = d_{i-1} + 8d_{i-2} + 4d_{i-3} - \frac{2^{i+2}-(-1)^i}{3}, \quad i \geqslant 3, \\ d_0 = 1 \qquad d_1 = 4 \qquad d_2 = 11. \end{gathered} \tag{20}$$

By standard techniques it can be shown that the solution to (20) is

$$d_i = \frac{17+7\sqrt{17}}{68}\left(\frac{3+\sqrt{17}}{2}\right)^i + \frac{17-7\sqrt{17}}{68}\left(\frac{3-\sqrt{17}}{2}\right)^i + \frac{2^{i+2}-(-1)^i}{6}. \tag{21}$$

The total number of entries in the first 2^i rows of the rhombus is 4^i. Since $(3+\sqrt{17})/8$ is approximately 0.89, we have the following result.

Theorem 2. *Let a_n and f_n be the number of ones and zeroes, respectively, in the first n rows of Pascal's rhombus* (mod 2)*. Then $\lim_{n\to\infty} a_n/f_n = 0$. If $n=2^i$ then $a_n = d_i$ where d_i is given by Eq.* (21) *and the fraction of ones in the first 2^i rows is asymptotically equal to* $(17+7\sqrt{17})/68((3+\sqrt{17})/8)^i$.

So taking 64 times as many rows cuts the fraction about in half.

Pascal's rhombus (mod 2) is uniquely determined by the recurrence relation (1) and the initial conditions specifying the first two rows. As suggested by Ľubomír Šoltés, we can use any $(0,1)$ first row S (and all zeroes in the previous row) and then use (1) to generate an array $\mathscr{U}(\mathscr{S})$. If S has N ones (so that S can be thought of as a finite sequence of zeroes and ones with two infinite tails of zeroes), then the array $\mathscr{U}(\mathscr{S})$ can be obtained by adding N copies of Pascal's rhombus (mod 2) with appropriate horizontal shifts according to S. This gives the following corollary.

Corollary 3. *Let S be a finite sequence of zeroes and ones. Let g_n and h_n be the number of ones and zeroes, respectively, in the first n rows of $\mathscr{U}(\mathscr{S})$. Then* $\lim_{n\to\infty} g_n/h_n = 0$.

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References

[1] J. Goldwasser, W. Klostermeyer, G. Trapp, Characterizing switch-setting problems, Linear Multilinear Algebra 43 (1997) 121–135.

[2] W. Klostermeyer, M. Mays, L. Soltes, G. Trapp, A Pascal rhombus, Fibonacci Quart. 35 (1997) 318–328.